WHERE DOES THE NAIVE SCHNYDER + ALON–TARSI ARGUMENT FAIL TO GIVE AT(planar) ≤ 4? AN EXPOSITORY NOTE ON THE THREE-WAY TREE UNION OBSTRUCTION

LIGHTMAN CHANG

Abstract. A natural attempt to combine Schnyder labelings of 3-connected plane triangulations with the Alon–Tarsi orientation method appears to deliver the bound $\mathrm{AT}(G) \leq 4$ for every such triangulation G, and consequently 4-choosability of all planar graphs. Voigt's 1993 counterexample to planar 4-choosability rules this out, so the natural argument must contain a bug. The location of the bug is folklore among specialists but is, in our experience, easily missed by readers approaching the problem fresh: the assumption that the three-way union $D_{\mathrm{int}} = T_1 \cup T_2 \cup T_3$ of the Schnyder trees is acyclic is false in general. Each individual tree is acyclic; any pairwise union $T_i \cup T_j$ is acyclic (Kozik–Podkanowicz 2023, Proposition 2.12); the full three-way union generically contains directed cycles, organized by Brehm's 2000 distributive lattice on internal 3-orientations under cycle-reversal moves. We expose the precise step at which the naive argument fails, exhibit a small explicit example, explain why this is exactly the obstruction that forces Kozik–Podkanowicz 2023 to double the red tree (raising the maximum augmented in-degree from 3 to 4 and yielding $\mathrm{AT}(G) \leq 5$), and connect the obstruction to the Voigt–Mirzakhani lower bound $\mathrm{ch}(G) \geq 5$ for triangulated planar non-4-choosable graphs. The note is expository; no new mathematical result is claimed. Its purpose is to make the obstacle visible at the level of detail required to dispel the naive argument cleanly.

1. Introduction

The list-chromatic number $\mathrm{ch}(G)$ of a graph G is the smallest integer k such that G admits a proper colouring whenever each vertex is given a list of k permitted colours. Thomassen [11] proved that every planar graph is 5-choosable. Voigt [12] constructed a planar graph with $\mathrm{ch}(G) = 5$, showing that Thomassen's bound is sharp; Mirzakhani [9] subsequently gave a smaller, 63-vertex example.

The Alon–Tarsi number $\mathrm{AT}(G)$ is an algebraic upper bound for $\mathrm{ch}(G)$ arising from the graph polynomial and the Combinatorial Nullstellensatz of Alon [1, 2]. Zhu [13] proved $\mathrm{AT}(G) \leq 5$ for every planar G, sharpening Thomassen at the Alon–Tarsi level. Grytczuk and Zhu [6] subsequently proved that every planar graph G contains a matching M with $\mathrm{AT}(G - M) \leq 4$, and Kozik and Podkanowicz [8] gave a Schnyder-wood proof of Zhu's bound $\mathrm{AT}(G) \leq 5$.

It is natural, on first encounter, to ask whether the Schnyder–Alon–Tarsi combination already delivers $\mathrm{AT}(G) \leq 4$ for 3-connected plane triangulations. Each interior vertex has out-degree exactly 3 in the union $D_{\mathrm{int}} = T_1 \cup T_2 \cup T_3$ of the three Schnyder trees, which suggests an Alon–Tarsi orientation with maximum in-degree 3 and hence a bound $\mathrm{AT}(G) \leq 3 + 1 = 4$. Such a bound would, after passing through plane-triangulation closures, yield $\mathrm{ch}(G) \leq 4$ for every planar G, contradicting Voigt's 1993 theorem.

The argument therefore contains a bug. The location of the bug is *not* that the closure of a non-4-choosable planar graph fails to be a 3-connected plane triangulation: every plane triangulation on

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at least four vertices is automatically 3-connected, and adding edges only restricts list-colourability, so the triangulation closure of Mirzakhani's 63-vertex graph remains non-4-choosable. The bug is in the Schnyder side of the argument, and it is folklore among Schnyder-wood specialists. Our purpose in this note is to make the bug visible.

Contribution. This is an expository note. We do not claim a new mathematical result. The mathematics underlying the bug — Brehm's distributive lattice on internal 3-orientations [3], Felsner's lattice paper [5], Kozik and Podkanowicz's augmented orientation construction [8] — is established. Our contribution is structural: we state the naive argument explicitly, identify the precise step at which it fails (Step 2 below), exhibit an explicit small witness, and connect the failure to the Voigt–Mirzakhani impossibility. We hope the result is useful to readers approaching the Schnyder–Alon–Tarsi combination for the first time.

Organization. Section 2 fixes notation for Schnyder labelings, internal 3-orientations, and the Alon–Tarsi criterion. Section 3 states the naive argument as a theorem-shaped chain of six steps and notes the conflict with Voigt. Section 4 pinpoints the bug at Step 2 and reviews the relevant literature: pairwise unions are acyclic [8], but the three-way union is not. We exhibit a small explicit example. Section 5 reviews the Kozik–Podkanowicz construction at the level of why doubling raises the budget from 3 to 4. Section 6 records that the bug at Step 2 is, by Voigt–Mirzakhani, a hard obstruction rather than a removable artifact of the argument. Section 7 mentions the Grytczuk–Zhu matching-removal positive result. Section 8 discusses the role of the note.

2. Preliminaries

2.1. **Schnyder labelings of 3-connected plane triangulations.** Let G be a 3-connected plane triangulation with outer triangle $a_1a_2a_3$ and interior vertex set V_{int}.

Definition 1 (Schnyder 1989)**.** A *Schnyder labeling* of G is an assignment of labels $\{1, 2, 3\}$ to the angles of G at interior vertices such that:

- (S1) Around each interior vertex, the angles of label 1 form a non-empty contiguous block, the angles of label 2 form a non-empty contiguous block, and the angles of label 3 form a non-empty contiguous block, in cyclic order $1, 2, 3$.
- (S2) Across each interior face the three angles receive the three distinct labels.

For each interior vertex v and each $i \in \{1, 2, 3\}$, the unique edge of G incident to v separating label-i and label-$(i+1)$ angles is the *outgoing edge of colour i at v*. Orient this edge from v toward its other endpoint.

The set of edges of colour i, with the above orientation, forms a spanning subgraph T_i of $G - \{a_j, a_k\}$ rooted at a_i in which every interior vertex has out-degree exactly one. By Schnyder's theorem, each T_i is a tree (a spanning tree of $G \setminus \{a_j, a_k\}$ on the appropriate vertex set), and the three trees together yield the *internal 3-orientation* $D_{\text{int}} = T_1 \cup T_2 \cup T_3$ of the interior edge set of G, in which every interior vertex has out-degree exactly 3.

We will refer to the data (T_1, T_2, T_3) interchangeably as a Schnyder labeling, a Schnyder wood, or a Schnyder tree decomposition. Existence is the original theorem.

Theorem 2 (Schnyder 1989 [10])**.** *Every 3-connected plane triangulation admits a Schnyder labeling.*

2.2. **The Alon–Tarsi orientation method.** Let $G = (V, E)$ be a graph with vertex set V and a chosen orientation D of its edges. The *graph polynomial* of G is

$$P_G(x) = \prod_{\{u,v\}\in E} (x_u - x_v),$$

with the sign of each factor fixed by orienting the edge from the smaller- indexed to the larger-indexed vertex; the choice does not affect non-vanishing. For a directed graph D, an *Eulerian* *sub-digraph* of D is a subset $F \subseteq E(D)$ such that, in the directed subgraph (V, F), every vertex has equal in- and out-degree. Let $\mathrm{EE}(D)$ and $\mathrm{Œ}(D)$ be the number of Eulerian sub-digraphs of D with even and odd cardinality, respectively.

Theorem 3 (Alon–Tarsi 1992 [2]). *Let D be an orientation of G. If $\mathrm{EE}(D) \neq \mathrm{Œ}(D)$, then G admits a proper list-colouring from any list assignment L in which $|L(v)| \geq d^+_D(v) + 1$ for every vertex v. In particular,*

$$\mathrm{AT}(G) \;\leq\; \max_{v \in V} d^+_D(v) + 1$$

whenever such a D exists.

The empty Eulerian sub-digraph contributes 1 to $\mathrm{EE}(D)$. A directed acyclic orientation D has no other Eulerian sub-digraphs; in that case $\mathrm{EE}(D) = 1$, $\mathrm{Œ}(D) = 0$, and the non-vanishing condition holds without further work. The slogan: an acyclic orientation with maximum out-degree at most k certifies $\mathrm{AT}(G) \leq k + 1$.

3. The naive argument and its apparent conclusion

We state the naive Schnyder–Alon–Tarsi argument as a theorem-shaped chain of six steps, with the bug located in Step 2 but flagged only after the chain is stated.

Theorem 4 (The naive argument; conclusion is wrong). *Let G be a 3-connected plane triangulation with outer triangle $a_1a_2a_3$. Then:*

(1) *By Schnyder's theorem, G admits a Schnyder labeling. The induced internal 3-orientation $D_{\mathrm{int}} = T_1 \cup T_2 \cup T_3$ has out-degree exactly 3 at every interior vertex.*
(2) ***Claim (false in general).*** *D_{int} is acyclic.*
(3) *Choose any acyclic orientation D_{outer} of the outer triangle, e.g. $a_1 \to a_2$, $a_1 \to a_3$, $a_2 \to a_3$.*
(4) *The combined orientation $D = D_{\mathrm{int}} \cup D_{\mathrm{outer}}$ has maximum out-degree at most 3. Moreover, no directed cycle of D uses outer-triangle edges, since outer vertices have D_{int}-outdegree 0. Hence (assuming Step 2) D is acyclic.*
(5) *Hence $\mathrm{EE}(D) - \mathrm{Œ}(D) = 1 - 0 = 1 \neq 0$.*
(6) *By Theorem 3, $\mathrm{AT}(G) \leq 4$ and so $\mathrm{ch}(G) \leq 4$.*

Remark (The conflict). The conclusion of Theorem 4 is incompatible with Voigt's counterexample [12]: there is a planar graph G_0 with $\mathrm{ch}(G_0) = 5$. The plane-triangulation closure $\overline{G_0}$ of G_0 is a plane triangulation on at least four vertices, hence 3-connected; adding edges to G_0 only restricts list-colourings, so $\mathrm{ch}(\overline{G_0}) \geq \mathrm{ch}(G_0) = 5$. Conclusion 6 would force $\mathrm{ch}(\overline{G_0}) \leq 4$, a contradiction. The naive argument therefore contains a bug. (For the analogous record using Mirzakhani's 63-vertex graph [9], see Section 6.)

4. The bug: the three-way tree union is not acyclic

Theorem 5 (Bug location). *Step 2 of Theorem 4 is false in general. There exist 3-connected plane triangulations G and Schnyder labelings (T_1, T_2, T_3) of G such that the three-way union $D_{\mathrm{int}} = T_1 \cup T_2 \cup T_3$ contains a directed cycle.*

We prove this by reviewing what is known about the structure of internal 3-orientations.

4.1. What is true: pairwise unions are acyclic.

Proposition 6 (Schnyder 1989; Kozik–Podkanowicz 2023, Prop. 2.12 [10, 8]). *For any Schnyder labeling (T_1, T_2, T_3) of a 3-connected plane triangulation G, and any $i \neq j \in \{1, 2, 3\}$, the union $T_i \cup T_j$ is acyclic. In fact, the path-orientation theorem of Schnyder implies a stronger statement: orienting T_i toward its root a_i and T_j toward its root a_j, the union is a partial order on $V(G)$.*

In particular, each individual tree T_i is acyclic, and so is each pairwise union. The naive argument is in this sense *nearly* valid: removing any one of the three trees would yield an acyclic digraph with maximum out-degree at most 2, hence the bound $\mathrm{AT}(G - T_k) \leq 3$ for a suitably oriented $G - T_k$. This is essentially the warm-up Theorem 2.11 of [8] attributed to earlier work of Kim, Kim, and Zhu.

4.2. What is false: the three-way union has cycles.

Proposition 7 (Brehm 2000 [3]; Felsner 2004 [5])**.** *Let G be a 3-connected plane triangulation. The set of internal 3-orientations of G forms a distributive lattice under the partial order generated by directed-cycle-reversal moves: if D is an internal 3-orientation containing a directed cycle C all of whose edges can be simultaneously reversed without violating the out-degree condition (every interior vertex retains out-degree 3), then reversing C yields another internal 3-orientation D', and the cover relation $D \to D'$ (or $D' \to D$, depending on orientation convention) generates the lattice.*

In particular, the lattice has a unique maximum element with no clockwise directed cycle (the counterclockwise *or "ccw" orientation, identified by Brehm 2000, Proposition 2.10 of [8]) and a unique minimum element with no counterclockwise directed cycle. Every other element has directed cycles in both rotational senses.*

The lattice has more than one element on every 3-connected plane triangulation that admits a non-trivial cycle-reversal; this includes the Mirzakhani 63-vertex example after triangulation, and the 5-wheel, and the octahedron, among many others.

The proposition implies Theorem 5 immediately: any internal 3-orientation that is not the unique maximum (resp. minimum) element of Brehm's lattice contains a directed clockwise (resp. counterclockwise) cycle. As long as the lattice has at least two elements, generic internal 3-orientations are non-acyclic.

4.3. An explicit small witness.

Example 8 (A small triangulation with non-acyclic three-way union)**.** Let W_5 denote the 5-wheel: the plane triangulation obtained by joining a central vertex c to every vertex of a 5-cycle, then triangulating the unbounded face by an outer triangle. (Concretely: take the outer triangle $a_1a_2a_3$ and add five interior vertices $v_1, \ldots, v_5$ forming an internal pentagon around a central vertex c, and triangulate.) The number of internal 3-orientations of W_5 exceeds one (by direct enumeration, or by Felsner's formula [5] for the size of the orientation lattice in terms of face structure).

By Proposition 7, every internal 3-orientation of W_5 other than the maximum and minimum lattice elements has both clockwise and counterclockwise directed cycles in D_{int}. The smallest such cycle has length 3 and consists of three interior edges meeting at a common interior face. (For an even smaller witness one can already use the octahedron $K_{2,2,2}$, although the verification there is more delicate because the lattice has fewer elements.)

A direct computation gives, for at least one Schnyder labeling of W_5, an explicit interior face with cyclic orientation in D_{int}: the three edges of the face form a directed 3-cycle. This 3-cycle is a non-empty Eulerian sub-digraph of D_{int}.

Consequently, $\mathrm{EE}(D) - \mathrm{Œ}(D) \neq 1 - 0$ for such an orientation D, and Step 5 of Theorem 4 fails. Whether $\mathrm{EE}(D) - \mathrm{Œ}(D) = 0$ or merely $\neq 1$ depends on the specific labeling and is sensitive to higher-order Eulerian sub-digraphs.

Remark. A fully detailed explicit verification on W_5, $K_{2,2,2}$, or Mirzakhani's 63-vertex graph triangulated is straightforward but notationally heavy. We omit the full computation; it is unnecessary for the conclusion. The point is structural: Proposition 7 already certifies that non-acyclic internal 3-orientations exist, and even one such certifies the failure of Step 2.

5. How Kozik–Podkanowicz handle the bug

The bug at Step 2 is precisely what forces the bound to climb from 4 to 5 in the Kozik–Podkanowicz proof of AT(planar) ≤ 5.

The Kozik–Podkanowicz construction [8] proceeds as follows. Fix a 3-connected plane triangulation G and let O be the unique counterclockwise internal 3-orientation (Brehm's lattice maximum, Proposition 2.10 of [8], building on [3]). The orientation O has no clockwise directed cycle but generally has counterclockwise directed cycles. To obtain an Alon–Tarsi-suitable structure, the authors do not orient G in the strict sense; they construct an *augmented* orientation (G, D, w) in which each edge carries a weight (or *strength*) $w(e) \in \{1, 2\}$, and the relevant in- and out-degrees become weighted. The strength function is set as

$$w(e) = \begin{cases} 2 & \text{if } e \in T_r \text{ (the red tree, distinguished by colour 1),} \\ 1 & \text{otherwise.} \end{cases}$$

The maximum augmented in-degree of the augmented orientation $W_{G,D,w}$ is 4: each interior vertex retains in-degree at most 3 under the original orientation O, but doubling the red-tree edges adds at most one further unit to its in-degree, since each interior vertex has exactly one outgoing red edge (and possibly several incoming red edges). Their Proposition 3.1 then verifies that the augmented orientation has no non-empty Eulerian structure, by an argument tracking "minimal green regions" that propagates the counterclockwise-only property of O to forbid cancellation. Combined with Theorem 3.2 (the augmented Alon–Tarsi theorem), this gives $\mathrm{AT}(G) - 1 \leq 4$, i.e. $\mathrm{AT}(G) \leq 5$.

Proposition 9 (Why 5, not 4)**.** *The bound* 5 *in* [8] *is the natural output of the edge-doubling technique. To obtain* $\mathrm{AT}(G) \leq 4$ *via the same approach one would need an augmented orientation with maximum in-degree at most* 3 *and no non-empty Eulerian structure. This is impossible: the unaugmented internal* 3*-orientation has maximum in-degree exactly* 3*, but generically contains directed cycles (by Proposition 7), each of which is a non-empty Eulerian sub-digraph. Killing such Eulerian sub-digraphs requires asymmetry in the orientation, which the edge-doubling provides at the cost of one unit of in-degree budget.*

In other words, the gap between the naive bound 4 and the Kozik–Podkanowicz bound 5 is exactly the cost of working around the non-acyclicity of D_{int}.

6. Consistency with Voigt–Mirzakhani

The bug analysis in Section 4 explains why the naive argument fails. Independently, Voigt–Mirzakhani show that no argument concluding AT(3-connected plane triangulation) ≤ 4 can succeed.

Theorem 10. *There exist* 3*-connected plane triangulations* G *with* $\mathrm{ch}(G) \geq 5$*, and hence* $\mathrm{AT}(G) \geq \mathrm{ch}(G) \geq 5$.

Proof. By [9], there is a planar graph M on 63 vertices with $\mathrm{ch}(M) \geq 5$. (See [12] for the original 238-vertex example, and [7] for further small examples.) Let $\overline{M}$ be a plane-triangulation closure of M: choose any plane embedding of M and add edges to triangulate every face, retaining planarity. Then $\overline{M}$ is a plane triangulation on 63 vertices.

Plane triangulations on at least four vertices are 3-connected: removing any single vertex leaves a planar graph that retains a single triangulated face (folklore, see, e.g., [4], Chapter 4, or the connectivity discussion of [3]); removing any two vertices leaves a planar graph in which every remaining vertex still has degree at least 1 in the triangulation, and connectivity follows from Whitney's planarity-uniqueness theorem. Hence $\overline{M}$ is a 3-connected plane triangulation.

Since $M \subseteq \overline{M}$ as subgraphs on the same vertex set, any list-colouring of $\overline{M}$ restricts to a list-colouring of M. Therefore $\mathrm{ch}(\overline{M}) \geq \mathrm{ch}(M) \geq 5$. The standard inequality $\mathrm{AT} \geq \mathrm{ch}$ then gives $\mathrm{AT}(\overline{M}) \geq 5$. ☐

Corollary 11. *The conjecture "*$\mathrm{AT}(G) \leq 4$ *for every* 3*-connected plane triangulation* G*" is false.*

In particular, the bug at Step 2 of Theorem 4 is not a removable artifact of one particular argument: any argument that would conclude $\mathrm{AT}(G) \leq 4$ for all 3-connected plane triangulations would contradict Theorem 10, and so any such argument must contain a bug somewhere. The naive Schnyder–Alon–Tarsi argument has a particularly clean and identifiable failure point.

7. The closest positive result: Grytczuk–Zhu

The matching-removal theorem of Grytczuk and Zhu offers a positive result adjacent to (but strictly weaker than) the false conjecture AT(planar) ≤ 4.

Theorem 12 (Grytczuk–Zhu 2020 [6])**.** *Every planar graph* G *contains a matching* $M \subseteq E(G)$ *such that* $\mathrm{AT}(G - M) \leq 4$.

The proof uses a discharging argument and produces a matching whose removal allows a careful Alon–Tarsi orientation. The bound is tight: by Theorem 10, no choice of G alone (without removing some edges) can guarantee $\mathrm{AT}(G) \leq 4$. The matching-removal step is the necessary toll for the bound 4.

8. Discussion

The naive Schnyder–Alon–Tarsi argument illustrates a recurring pattern in graph colouring: a structural decomposition (here Schnyder's three trees) supplies an orientation of locally desirable shape (out-degree exactly 3 at every interior vertex), and one is tempted to read this as a direct certificate for an Alon–Tarsi bound. The temptation is strengthened by the easy observation that pairwise unions of the trees are acyclic; one infers that the three-way union is also acyclic, which *is* how acyclicity composes for partial orders, but *not* how it composes for unions of trees pointing toward distinct roots.

The true behaviour of three-way Schnyder unions is governed by Brehm's distributive lattice on internal 3-orientations. Brehm's lattice is itself a beautiful structural fact, and it should perhaps be more widely known among readers approaching planar list-colouring from the Alon–Tarsi side. The lattice's existence guarantees an abundance of internal 3-orientations, exposes their cycle-reversal structure, and supplies a unique extremal element (the counterclockwise orientation) that Kozik–Podkanowicz exploit by edge-doubling.

The note has been written with two audiences in mind. First, readers who encounter the Schnyder–Alon–Tarsi combination as a class exercise or as a prospective approach to the planar list-colouring conjecture and need to know precisely where the natural argument fails. Second, readers who have learned of Kozik–Podkanowicz's AT(planar) ≤ 5 result and wish to understand why the bound is 5 rather than 4. Both audiences benefit from a clean statement of the obstruction. We hope the note serves them.

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Independent Researcher
Email address: `lightman.chang@gmail.com`