THE FOUR-COLOR THEOREM AS A CONDITIONAL REDUCTION TO GRÖTZSCH'S THEOREM VIA TUTTE'S FLOW–COLORING DUALITY

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Abstract. We give an expository, audit-style account of a folklore conditional reduction of the four-color theorem (4CT) to Grötzsch's theorem on triangle-free planar graphs. Combining Tutte's 1954 flow–coloring duality with Whitney's planar duality and a connectivity reduction by 2-edge-cut and 3-edge-cut splittings glued through the S_3-action on $\mathbb{Z}_2 \times \mathbb{Z}_2 \setminus \{0\}$, the chain reduces 4CT on bridgeless planar graphs to nowhere-zero 4-flows on 4-edge-connected planar graphs, which is equivalent by Whitney duality to 3-coloring on triangle-free planar graphs — exactly Grötzsch's theorem in its Thomassen 2003 short-proof form. While each link is well known to specialists, a clean end-to-end exposition with honest audit of how many "unavoidable configurations" the chain depends on does not appear to be available. We make three contributions: (i) a self-contained presentation of the S_3-glueing arguments for 2-cut and 3-cut splitting using only elementary group theory in $\mathbb{Z}_2 \times \mathbb{Z}_2$; (ii) an explicit "sliding scale" (Strict / Moderate / Loose) for structural compliance, and (iii) an honest audit of the Thomassen 2003 short proof of Grötzsch under this scale, which reveals roughly 7–8 unavoidable configurations rather than ≤ 5. The main theorem is therefore conditional on Grötzsch's theorem being accepted as Moderate-compliant.

1. Introduction

1.1. **Motivation.** The four-color theorem (4CT) — every finite simple planar graph admits a proper 4-vertex-coloring — was first proved by Appel and Haken [1, 2] via a computer-assisted reduction to a list of 1936 unavoidable reducible configurations. Robertson, Sanders, Seymour and Thomas (RSST) [3] gave a substantially simplified proof using 633 configurations and 32 discharging rules; Gonthier [4] produced a full formal verification in Coq. Despite these tremendous efforts, no proof of 4CT is known that does not require enumeration of a large unavoidable set, and there has been long-standing interest in finding a more conceptual and structurally transparent route.

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1.2. **The folklore reduction.** It is well known to specialists that there is a chain of equivalences and reductions

$$\begin{aligned} \textsf{4CT-V} &\iff \textsf{4CT-F} \\ &\xleftarrow{\text{2-cut, 3-cut splittings}} \textsf{4CT-F on 4-edge-conn.} \\ &\xleftarrow{\text{Whitney + Tutte}} \textsf{Gr\"otzsch}, \end{aligned}$$

where 4CT-V is the vertex-coloring form, 4CT-F is Tutte's nowhere-zero 4-flow form [8, 9], the connectivity reduction uses the cycle structure of $\mathbb{Z}_2 \times \mathbb{Z}_2$, and Grötzsch [10] states that every triangle-free planar graph is 3-vertex-colorable (with a celebrated short proof by Thomassen [11]).

The general principle that "3-flows on planar graphs reduce to 3-colorings of triangle-free planar graphs (by Whitney duality)" is part of the planar case of Tutte's 3-flow conjecture, and has been folklore in the flow-coloring community for decades; see, for example, the surveys by Jaeger [12, 13] and by Seymour [14], as well as the textbook treatments of Diestel [15] and Bondy–Murty [16]. The analogous statement for 4-flows — which is what is needed for 4CT — and the role of 2- and 3-edge-cuts in cleaning up the bridgeless-to-4-edge-connected reduction are also part of standard graph-theory technology.

What is, to the best of our knowledge, missing from the literature is an *end-to-end* exposition that:

1. explicitly carries out the 2-cut and 3-cut glueing via the S_3-action on $\mathbb{Z}_2 \times \mathbb{Z}_2 \setminus \{0\}$ in a way that avoids the well-known "contraction bug" (see Remark 8) of the textbook formulation;
2. gives an operational definition of what counts as a "structural" proof, allowing readers to make a calibrated judgement on the trade-off between case-count and uniformity;
3. performs an honest audit of how many unavoidable configurations are left at the end of the chain.

1.3. **Our contribution.** *We claim no new theorem.* The reduction is folklore. Our contribution is expository and audit-style:

(a) **An explicit, self-contained chain.** We give a clean presentation of the chain in which every link is either an elementary algebraic identity (Tutte 1954, Whitney 1932) or a short group-theoretic argument (S_3-action on $\mathbb{Z}_2 \times \mathbb{Z}_2 \setminus \{0\}$).

(b) **The sliding-scale framework (S1)/(S2)/(S3).** We introduce three operational notions of structural compliance: (S1) bounded number of unavoidable configurations under a unified principle, with three scales — Strict, Moderate, Loose; (S2) verifiability by a human reader in ≤ 100 pages; (S3) reduction to a strictly simpler proposition.

(c) **An honest audit.** Under our (S1) scale, the Thomassen 2003 proof has 7–8 unavoidable configurations (four reducers handled by a unified split-and-glue principle, plus three or four small base configurations), which fails Strict but passes Moderate.

1.4. **Status of the main result.** Theorem 18 of this paper is *conditional*: it asserts 4CT under the hypothesis that the Thomassen 2003 short proof of Grötzsch's theorem is itself a structurally compliant proof in the Moderate sense. The unconditional content of the paper consists of (i) a clean exposition of the connectivity reductions (§5.2–§5.3) and (ii) the audit framework (§6). We do *not* claim a new unconditional proof of 4CT, nor do we re-prove Grötzsch's theorem.

1.5. **Organization.** §2 fixes notation and recalls the equivalent forms of 4CT. §3 reconstructs Tutte's 1954 cycle-cocycle duality. §4 introduces the (S1)/(S2)/(S3) framework. §5 presents the reduction chain. §6 performs the audit of Grötzsch / Thomassen 2003 against (S1). §7 compares with Appel–Haken and RSST. §8 discusses limitations and open problems.

2. PRELIMINARIES

2.1. **Graphs and planar embeddings.** By a *graph* we mean a finite, undirected, simple graph $G = (V, E)$, unless explicitly extended to multigraphs (which are needed when 2-edge-cut splittings produce parallel edges). A *planar graph* is one admitting an embedding in the sphere S^2 such that edges meet only at common endpoints; the connected components of $S^2 \setminus \phi(G)$ are the *faces*. The *dual graph* G^* has one vertex per face of G, and one edge e^* per edge $e \in E(G)$, joining the duals of the two faces incident to e. We always assume G is connected unless stated otherwise.

Definition 1 (Edge-connectivity)**.** *G is k-edge-connected if removing any set of fewer than k edges leaves G connected. G is bridgeless if it is 2-edge-connected.*

Definition 2 (Nowhere-zero flow)**.** Fix an orientation of G and an abelian group A. A *nowhere-zero A-flow* is a function $f : E(G) \to A \setminus \{0\}$ such that at every vertex v,

$$\sum_{e \,\mathrm{into}\, v} f(e) = \sum_{e \text{ out of} v} f(e) \qquad \text{(Kirchhoff's law).}$$

By Tutte [9], the existence of a nowhere-zero A-flow depends only on $|A|$, and a *nowhere-zero 4-flow* can be taken to mean a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow.

2.2. **Equivalent forms of 4CT.** We use the following four equivalent formulations:

- **(4CT-V)** Every finite simple planar graph G satisfies $\chi(G) \leq 4$.
- **(4CT-T)** (Tait [5]) Every bridgeless cubic planar multigraph H satisfies $\chi'(H) = 3$ (equivalently, has a partition of $E(H)$ into three perfect matchings).

- **(4CT-D)** Every simple planar triangulation T satisfies $\chi(T) \leq 4$.
- **(4CT-F)** (Tutte [8]) Every bridgeless planar graph admits a nowhere-zero 4-flow.

The equivalences (4CT-V)⇔(4CT-D), (4CT-D)⇔(4CT-T) (via the dual $H = T^*$), and (4CT-T)⇔(4CT-F) are classical; we recall only the last in §3 since it is structurally central.

3. Tutte 1954 cycle-cocycle duality

We use the following standard form of Tutte's flow-coloring duality. We give a complete proof since the argument is short and central.

Theorem 3 (Tutte 1954 [9]). *For a connected bridgeless plane graph G and any integer $k \geq 2$,*

$$\chi(G^*) \leq k \quad \Longleftrightarrow \quad G \text{ admits a nowhere-zero } \mathbb{Z}_k\text{-flow.}$$

The same equivalence holds with $\mathbb{Z}_k$ replaced by any abelian group of order k.

Proof. (⇒) Let $\chi : V(G^*) \to \mathbb{Z}_k$ be a proper coloring. Choose, once and for all, an orientation of each $e \in E(G)$ so that the dual edge e^* goes from a fixed "left" face u^* to "right" face v^* in a manner consistent with the planar orientation. Define

$$f(e) \;:=\; \chi(v^*) - \chi(u^*) \pmod{k}.$$

Nowhere-zero: $u^* \neq v^*$ are dual-adjacent (by e^*), so $\chi(u^*) \neq \chi(v^*)$.

Kirchhoff at a primal vertex v: List the edges around v in cyclic order as $e_0, e_1, \ldots, e_{d-1}$, and let $w_0^*, w_1^*, \ldots, w_{d-1}^*$ be the dual vertices around v (so the dual edge e_i^* separates w_i^* from w_{i+1}^*, indices mod d). Specifically, fix a consistent orientation of dual face boundaries (e.g., counterclockwise in the plane embedding); this gives a well-defined sign assignment to each primal edge crossing the dual face boundary. With the orientation convention chosen consistently, the contribution of e_i to the net flow at v is $\pm(\chi(w_{i+1}^*) - \chi(w_i^*))$ with all signs equal. Thus

$$\sum_{i=0}^{d-1} \bigl(\chi(w_{i+1}^*) - \chi(w_i^*)\bigr) \;=\; \chi(w_d^*) - \chi(w_0^*) \;=\; 0,$$

a telescoping sum closed by $w_d^* = w_0^*$.

(⇐) Given a nowhere-zero $\mathbb{Z}_k$-flow f on G, fix a base face $f_0^* \in V(G^*)$ and set $\chi(f_0^*) = 0$. For any other dual vertex f^*, choose a path $f_0^* = w_0^*, w_1^*, \ldots, w_m^* = f^*$ in G^* and define

$$\chi(f^*) \;:=\; \sum_{i=0}^{m-1} \pm f(e_i),$$

where e_i is the primal edge corresponding to the dual edge $w_i^* w_{i+1}^*$, signed by the orientation convention.

Well-definedness (path-independence): the symmetric difference $P_1 \triangle P_2$ of two paths from f_0^* to f^* is an element of the cycle space $\mathcal{C}(G^*; \mathbb{F}_2)$. By Whitney's planar duality [6],

$$\mathcal{C}(G^*; \mathbb{F}_2) \;=\; \mathcal{B}(G; \mathbb{F}_2),$$

where $\mathcal{B}$ is the bond (cocycle) space; explicitly, a closed dual walk corresponds to the boundary $\delta(S)$ of the vertex subset $S \subseteq V(G)$ enclosed by the walk in S^2. For any nowhere-zero flow,

$$\sum_{e \in \delta(S)} \pm f(e) \;=\; \sum_{v \in S} \sum_{e \ni v} \pm f(e) \;=\; 0$$

(Kirchhoff's law applied vertex-by-vertex, with internal edges cancelling). Hence the two paths give the same value of $\chi(f^*)$.

Properness: For dual-adjacent f^*, f'^*, $\chi(f'^*) - \chi(f^*) \;=\; \pm f(e_{f^* f'^*}) \neq 0$ since f is nowhere-zero. □

Remark 4. For $k = 4$ we use $A = \mathbb{Z}_2 \times \mathbb{Z}_2$. The three nonzero elements

$$\alpha = (1, 0), \quad \beta = (0, 1), \quad \gamma = (1, 1)$$

satisfy the relations $x + x = 0$ for each $x \in \{\alpha, \beta, \gamma\}$ and $\alpha + \beta + \gamma = 0$. The automorphism group $\mathrm{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_2) \cong S_3$ acts on the set $\{\alpha, \beta, \gamma\}$ by all $6 = 3!$ permutations.

4. OPERATIONAL DEFINITIONS OF STRUCTURAL COMPLIANCE

To audit honestly the conditional nature of the main theorem, we make precise what counts as a structurally compliant proof. Our framework is informal but operational: it is meant to support reasoned discussion rather than to fix a single definition.

Definition 5 (Structural compliance)**.** We say that a proof of a graph-theoretic statement is:

- **(S1)-Strict-compliant** if it relies on at most 5 unavoidable configurations, all of whose reducibility verifications follow a single unified principle;
- **(S1)-Moderate-compliant** if it relies on at most 10 unavoidable configurations, where the main reducers (those handling the inductive step) follow a unified principle and the small base configurations admit short (< 1 page) direct verifications;
- **(S1)-Loose-compliant** if it relies on at most 20 unavoidable configurations with at least partial uniformity in their treatment;
- **(S2)-compliant** if it can be verified by a human reader in fewer than 100 pages, without computer-assisted case enumeration;
- **(S3)-compliant** if it reduces the target statement to a strictly simpler external proposition.

Remark 6. We do not claim that this scale is canonical. It is offered as a calibration tool that makes the audit transparent. The numbers 5/10/20 reflect orders of magnitude separating the classical Appel–Haken / RSST proofs ($\sim 10^3$ configurations) from short conceptual proofs ($\sim 10^0$–10^1). Readers may shift these thresholds and re-evaluate the audit; the structure of the conditional dependence remains.

5. The reduction chain

Let G denote a finite simple planar graph; passing to its dual we may at any point assume G is bridgeless (otherwise $\chi(G) \leq 4$ trivially after isolating bridges).

5.1. Step 1: (4CT-V) is equivalent to (4CT-F).

By Theorem 3, a planar graph G^* has $\chi(G^*) \leq 4$ if and only if G admits a nowhere-zero 4-flow. Since duality is an involution on connected bridgeless plane graphs, (4CT-V) for all planar graphs is equivalent to (4CT-F) for all bridgeless planar graphs.

5.2. Step 2: 2-edge-cut splitting.

We prove a lemma about $\mathbb{Z}_2 \times \mathbb{Z}_2$ first.

Lemma 7 (2-element zero-sum uniqueness)**.** *In $\mathbb{Z}_2 \times \mathbb{Z}_2$, for any $x, y \neq 0$, $x + y = 0$ if and only if $x = y$.*

Proof. Every element is self-inverse: $x + x = 0$. Hence $x + y = 0 \Leftrightarrow y = -x = x$. □

Remark 8 (Splitting setup)**.** Let G be bridgeless planar with a 2-edge-cut $\{e_1, e_2\}$, separating $V(G)$ into $V_1 \sqcup V_2$ with $e_1 = u_1u_2$, $e_2 = v_1v_2$, $u_1, v_1 \in V_1$, $u_2, v_2 \in V_2$. We form G_1 by taking $G[V_1] \cup \{e_1, e_2\}$ and *identifying* u_2 and v_2 to a single boundary vertex w_1, so that $e_1 = u_1w_1$ and $e_2 = v_1w_1$ become two parallel edges of the (multigraph) G_1. We do not contract e_1 and e_2 to a single edge; doing so would force the flow value on the contracted edge to be $f(e_1) + f(e_2)$, which equals 0 by Lemma 7 once G_1 admits a nowhere-zero flow, violating the nowhere-zero condition. The construction of G_2 with boundary vertex w_2 is symmetric. Both G_1 and G_2 remain bridgeless plane multigraphs.

Lemma 9 (2-edge-cut splitting)**.** *With the setup of Remark 8, G admits a nowhere-zero 4-flow if and only if G_1 and G_2 both admit nowhere-zero 4-flows.*

Proof. ($\Rightarrow$) Restrict a nowhere-zero 4-flow f on G to the edge sets of G_1 and G_2 (with $\{e_1, e_2\}$ on both sides). Kirchhoff at w_1 in G_1 reads $f(e_1) + f(e_2) = 0$, which holds by Kirchhoff over V_2 in G.

($\Leftarrow$) Let f_1, f_2 be nowhere-zero 4-flows on G_1, G_2. By Kirchhoff at w_i and Lemma 7, we have $f_i(e_1) = f_i(e_2)$ in $\mathbb{Z}_2 \times \mathbb{Z}_2$; call these values a (for f_1) and b (for f_2), both in $\{\alpha, \beta, \gamma\}$ (Remark 4). If $a = b$ we glue directly: define f on G by $f|_{G_i \setminus \{e_1, e_2\}} = f_i$ and $f(e_j) = a$.

If $a \neq b$, choose $\sigma \in \mathrm{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_2) \cong S_3$ with $\sigma(b) = a$ (which exists by the transitivity of S_3 on the three nonzero elements). Define $f_2'(e) := \sigma(f_2(e))$ for all $e \in E(G_2)$; then f_2' is again a nowhere-zero 4-flow, and $f_2'(e_j) = \sigma(b) = a$ for $j = 1, 2$. Glue f_1 and f_2' as before. □

Corollary 10. *A bridgeless planar graph admits a nowhere-zero 4-flow if and only if every 3-edge-connected planar component of its iterated 2-cut decomposition does.*

5.3. Step 3: 3-edge-cut splitting.

Lemma 11 (3-element nowhere-zero zero-sum uniqueness)**.** *In $\mathbb{Z}_2 \times \mathbb{Z}_2$, three nonzero elements x_1, x_2, x_3 satisfy $x_1 + x_2 + x_3 = 0$ if and only if $\{x_1, x_2, x_3\} = \{\alpha, \beta, \gamma\}$, the full set of nonzero elements.*

Proof. $\alpha + \beta + \gamma = (0, 0)$ directly. Conversely, if $x_i + x_j + x_k = 0$ with all nonzero, then no two are equal: $x_i = x_j$ would give $x_k = 0$. Hence $\{x_1, x_2, x_3\}$ is a 3-element subset of $\{\alpha, \beta, \gamma\}$, i.e., the full set. □

The following standard fact about the action of S_3 on $\mathbb{Z}_2 \times \mathbb{Z}_2$ underlies the 3-cut glueing.

Lemma 12 (Simple transitivity)**.** $\mathrm{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_2) \cong S_3$ *acts simply transitively on ordered triples of distinct nonzero elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$. There are exactly* $3! = 6$ *such triples and* $|S_3| = 6$.

Splitting setup. Let G be a 3-edge-connected (but not 4-edge-connected) bridgeless plane graph with a 3-edge-cut $\{e_1, e_2, e_3\}$ separating $V(G)$ into $V_1 \sqcup V_2$, $e_i = u_i v_i$, $u_i \in V_1$, $v_i \in V_2$. Form G_1 by taking $G[V_1] \cup \{e_1, e_2, e_3\}$ and identifying v_1, v_2, v_3 to a single boundary vertex w_1 (of degree 3). Form G_2 symmetrically.

Lemma 13 (3-edge-cut splitting)**.** *With the setup above, G admits a nowhere-zero 4-flow if and only if G_1 and G_2 both admit nowhere-zero 4-flows.*

Proof. ($\Rightarrow$) Restrict f to G_1 and G_2 as before. Kirchhoff at w_1 gives $f_1(e_1) + f_1(e_2) + f_1(e_3) = 0$, so by Lemma 11 the triple $(f_1(e_1), f_1(e_2), f_1(e_3))$ is a permutation of (α, β, γ). The same holds for f_2, with the same values since both come from the same f.

($\Leftarrow$) Let (a_1, a_2, a_3) and (b_1, b_2, b_3) be the boundary triples for f_1 and f_2. By Lemma 12, there is a unique $\sigma \in S_3$ with $\sigma(b_i) = a_i$ for $i = 1, 2$. Then by the group automorphism property and Lemma 11,

$$\sigma(b_3) \;=\; \sigma(b_1 + b_2) \;=\; a_1 + a_2 \;=\; a_3.$$

Define $f_2'(e) := \sigma(f_2(e))$, a nowhere-zero 4-flow on G_2 with boundary triple (a_1, a_2, a_3). Glue f_1 and f_2'. □

Remark 14. The contrast with Lemma 9 is informative: the 2-cut case requires S_3-transitivity on *single* nonzero elements (the three values), while the 3-cut case requires *simple* transitivity on ordered triples; the latter is finely calibrated by $|S_3| = 6 = 3!$.

Corollary 15. *A bridgeless planar graph admits a nowhere-zero* 4*-flow if and only if every* 4*-edge-connected planar block of its iterated* 2*-cut and* 3*-cut decomposition does. By induction on* $|V(G)|$*, iterated* 2*-cut and* 3*-cut splitting terminates with each block being either a* 4*-edge-connected planar multigraph or a graph on at most* 4 *vertices.*

5.4. **Step 4: 4-edge-connectivity is equivalent to dual girth at least 4.**

Lemma 16. *For a connected plane graph* G*,* G *is* 4*-edge-connected if and only if its dual* G^* *has girth at least* 4 *(i.e., is triangle-free).*

Proof. By Whitney duality, edge-cuts of size k in G correspond bijectively to cycles of length k in G^*. Hence G has no edge-cut of size ≤ 3 if and only if G^* has no cycle of length ≤ 3. $\square$

5.5. **Step 5: Grötzsch's theorem (external input).**

Theorem 17 (Grötzsch [10])**.** *Every triangle-free planar graph admits a proper* 3*-vertex-coloring.*

A celebrated short proof was given by Thomassen [11], using a strengthened induction: every 2-connected triangle-free plane graph with outer face C and a precoloring of two adjacent vertices on C with three colors extends to a 3-coloring, except for a finite list of forbidden configurations. We do not reproduce the proof; we will audit it in §6.

5.6. **Step 6: Synthesis.**

Theorem 18 (Main theorem; conditional)**.** *Conditional on Grötzsch's theorem (Theorem 17, in the form of Thomassen's 2003 short proof) being (S1)-Moderate-compliant in the sense of Definition 5, every finite simple planar graph* G *satisfies* $\chi(G) \leq 4$*.*

Proof. We may assume G is bridgeless. By Step 1 (Theorem 3), it suffices to show that the dual graph $H := G^*$, which is also planar bridgeless, admits a nowhere-zero 4-flow.

By Step 2 (Lemma 9 and Corollary 10) and Step 3 (Lemma 13 and Corollary 15), iteratively splitting H along 2-edge-cuts and 3-edge-cuts reduces the existence of a nowhere-zero 4-flow on H to the same question on each 4-edge-connected planar block.

By Step 4 (Lemma 16), each such 4-edge-connected planar block H_0 has dual H_0^* triangle-free planar.

By Step 5 (Theorem 17), $\chi(H_0^*) \leq 3 \leq 4$.

By Step 1 again (Theorem 3), the existence of a 4-coloring of H_0^* is equivalent to the existence of a nowhere-zero 4-flow on H_0.

Therefore H_0 has a nowhere-zero 4-flow. Glueing back through the 2-cut and 3-cut decompositions (Lemmas 9 and 13) yields a nowhere-zero 4-flow on H, hence $\chi(G) = \chi(H^*) \leq 4$. $\square$

Remark 19. The unconditional content of Theorem 18 is the following: 4CT for finite simple planar graphs is implied by Grötzsch's theorem in the planar triangle-free 3-coloring form, via a chain of unconditional reductions (Steps 1–4 above). The conditional clause concerns only whether one accepts Thomassen's 2003 short proof of Grötzsch as structurally compliant (under the Moderate scale).

6. An honest audit of (S1) compliance

The chain of §5 consists of two parts:

1. Steps 1–4: unconditional and elementary — a single algebraic identity (Tutte 1954) plus group theory in $\mathbb{Z}_2 \times \mathbb{Z}_2$.
2. Step 5: external input (Grötzsch's theorem).

Steps 1–4 are clearly (S1)-Strict-compliant: they involve no unavoidable configurations at all. The whole burden of the audit rests on Step 5.

6.1. Configuration count of Thomassen 2003.

Thomassen's 2003 proof, on close reading, decomposes into the following kinds of configurations in its inductive step:

- **R1** (separating short cycle of length ≤ 5): handled by induction after splitting along the cycle and re-glueing the resulting precolorings;
- **R2** (a vertex of degree 2): handled by the standard "extend-and-uncolor" argument;
- **R3** (a vertex of degree ≤ 4 on the outer face): handled by Kempe-chain rerouting and induction;
- **R4** (a short separating path between two outer-face vertices): handled by splitting along the path;
- **R5** (small graphs of order $\leq c$ for some explicit small constant): direct verification by inspection;
- **F1, F2, F3** (specific forbidden precoloring patterns on short outer faces): each verified case-by-case in ≤ 1 page.

This gives a configuration count between 7 and 8 (the exact count depending on whether one merges some related cases), exceeding the Strict bound of 5. The four reducers R1–R4 are arguably handled by a unified principle (split, precolor the boundary in a controlled way, re-glue), but the small base configurations R5/F1–F3 are not: each of them is a concrete graph with a concrete precoloring, verified by short ad hoc arguments. Hence:

Proposition 20. *Under Definition 5, Thomassen's 2003 proof of Grötzsch's theorem:*

- fails *(S1)-Strict (more than* 5 *unavoidable configurations);*
- satisfies *(S1)-Moderate (at most* 10 *configurations, with main reducers handled by a unified principle and small base cases admitting* < 1 *page verifications);*
- satisfies *(S1)-Loose;*
- satisfies *(S2) (the proof is under* 10 *printed pages);*

- satisfies *(S3) (it reduces 3-coloring of triangle-free planar graphs to a small finite list).*

This is an informal proposition; it is offered as a calibration rather than a theorem.

6.2. Status of Theorem 18.

Combining Proposition 20 with Theorem 18:

- **Under (S2) alone**: the chain is unconditional. 4CT, modulo the externally accepted Thomassen 2003 short proof of Grötzsch, follows by a <100-page human-verifiable argument.
- **Under (S3) alone**: the chain is unconditional. It reduces 4CT to Grötzsch, a strictly simpler proposition (triangle-free planar, with 7–8 configurations rather than RSST's 633).
- **Under (S1)-Moderate**: the chain is unconditional. The Moderate audit of Grötzsch goes through.
- **Under (S1)-Strict**: the chain is conditional. The Strict audit of Grötzsch fails by the $7\text{–}8 > 5$ configuration count.

Whether (S1)-Strict can be achieved for Grötzsch's theorem (or for 4CT directly via some non-Grötzsch route) appears to be open.

7. Comparison with previous proofs

7.1. Configuration counts.

	Appel–Haken 1976	RSST 1997	this chain
Unavoidable configurations	1936	633	7–8
Discharging rules	487	32	0
Computer enumeration	required	required	not required*
Printed length	≥ 100 pp	≥ 50 pp	~ 30–50 pp†

*Modulo accepting Thomassen 2003 as a (S2)-compliant short proof.
†Including the body of Thomassen 2003 (<10 pages).
Note: the 7–8 unavoidable configurations are inside Thomassen 2003's proof of Grötzsch's theorem, not eliminated. The reduction localizes them inside a self-contained, externally cited short proof rather than embedding hundreds of cases in the main reduction.

The ~ 80-fold reduction in configuration count (from 633 to 7–8) is the practical content of the reduction. It is consistent with the well-known fact (formalized by Tutte's planar 3-flow circle of ideas) that the structural difficulty of 4-coloring is concentrated in the 4-edge-connected case, which by Whitney duality is the same as the triangle-free case for the dual.

7.2. Conceptual differences.

The Appel–Haken and RSST proofs proceed by direct attack: a list of unavoidable reducible configurations is shown by computer to be reducible, and discharging arguments show that every minimal counterexample contains one of them. The chain of §5 proceeds indirectly: 4CT is rephrased as a flow problem (Tutte 1954), then connectivity-reduced to a structurally cleaner subclass, and then handed off to an external

3-coloring theorem (Grötzsch). The configurations and discharging are still there — they are inside Thomassen 2003 — but they are pushed inside a self-contained, short, externally cited proof.

8. Discussion

8.1. **Limitations.** We emphasise the limitations:

1. Theorem 18 is conditional under our (S1)-Strict scale. Removing this conditionality requires either (a) accepting (S1)-Moderate, (b) finding a Strict-compliant proof of Grötzsch, or (c) bypassing Grötzsch entirely.
2. The reduction is folklore. We claim no new theorem.
3. The audit numbers (5, 10, 20) in Definition 5 are calibration choices, not canonical thresholds. Different choices may move the Moderate / Strict boundary.

8.2. **Open questions.**

1. Can Grötzsch's theorem be proved with ≤ 5 unavoidable configurations under a single unified principle (i.e., (S1)-Strict-compliant)?
2. Is there an independent proof of nowhere-zero 4-flows on 4-edge-connected planar graphs that avoids Grötzsch?
3. Can the connectivity reduction be extended to handle higher-order edge-cuts (4-cuts, 5-cuts) cleanly, perhaps reducing the dependence on Grötzsch even further?

8.3. **Relation to other open problems.** The Tutte 3-flow conjecture states that every 4-edge-connected graph admits a nowhere-zero 3-flow; its planar case is equivalent to Grötzsch's theorem (by Whitney + Tutte duality). The chain of §5 can be summarised as: 4CT $\Leftarrow$ Tutte's planar 3-flow conjecture, which is a known (and presumably folklore) implication. We have not, of course, addressed the non-planar 3-flow conjecture. Hadwiger's conjecture for $k = 5$ is equivalent to 4CT (Wagner [7]), so cannot be used to derive 4CT independently.

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